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GRADIENT METHODS IN CONTROL THEORY
PART 1 - ORDINARY GRADIENT METHOD

by

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Abstract. An analytical approach to the gradient method is presented within the framework of the Bolza problem of the calculus of variations. The first variation is minimized subject to the linearized differential constraint and an isoperimetric constraint on the control variation. Since the resulting Euler equations are linear, the differential system describing the optimum corrections is linear. The properties of this system are studied, and the solutions are related to the stepsize α. Next, the optimization of α is performed by minimizing the sum of the first variation and the second variation; an analytical expression is derived for the optimum value of α. Thus, the present method is a hybrid, in that the shape of the system of variations is obtained from first-order considerations while the scale factor for the variations is obtained from second-order considerations. Two numerical examples illustrating the convergence properties of the algorithm are supplied.

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1. Introduction

Over the past decade, considerable work has been done on the application of gradient methods to control theory. In this area of research, the work of Bryson (Ref. 1) and Kelley (Ref. 2) has become renowned. Basically, the authors of Refs. 1-2 minimized a linearized functional subjected to a linearized differential constraint and a quadratic constraint on the system of control variations. A common characteristic of the Bryson-Kelley approach is that, in the derivation of the gradient algorithm, a preliminary integration of the linearized constraint is performed in order to obtain the state change $\delta x(t)$ in terms of the control change $\delta u(t)$. This integration is performed prior to optimizing the control change.

Although the above approach is correct, this preliminary integration is not necessary. In the opinion of this author, a much simpler derivation of the gradient algorithm is possible if one avoids integrating the state in terms of the control and views the minimal problem as a variational problem of the Bolza type with an added isoperimetric constraint of the quadratic type on the control variation.

After the system of variations is obtained from first-order considerations, second-order considerations are employed in order to determine the scale factor for the variations. Thus, the method presented here is a hybrid, in that it combines first-order considerations with second-order considerations.

2. Statement of the Problem

The purpose of this paper is to find the minimum of the functional

$$I = \int_{t_i}^{t_f} f(x, u, t)\, dt + [g(x)]_f \tag{1}$$

subject to the differential constraint

$$\dot{x} - \varphi(x, u, t) = 0 \tag{2}$$

and the end conditions

$$t_i = \text{given} \quad , \quad x_i = \text{given} \tag{3}$$

$$t_f = \text{given} \quad , \quad x_f = \text{free} \tag{4}$$

In the above equations, f and g are scalar functions; the vectors x, u, φ are respectively defined as follows:

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} \quad , \quad u = \begin{bmatrix} u^1 \\ u^2 \\ \vdots \\ u^m \end{bmatrix} \tag{5}$$

and

$$\varphi(x, u, t) = \begin{bmatrix} \varphi^1(x, u, t) \\ \varphi^2(x, u, t) \\ \vdots \\ \varphi^n(x, u, t) \end{bmatrix} \tag{6}$$

The time t is the independent variable; the state variable x and the control variable u are the dependent variables; the dot denotes the derivative with respect to the time. The subscript i refers to the initial point, and the subscript f refers to the final point.

The basic idea is to construct corrections $\delta x(t)$, $\delta u(t)$ leading from nominal functions $x(t)$, $u(t)$ to varied functions $\tilde{x}(t)$, $\tilde{u}(t)$ such that

$$\tilde{I} < I \tag{7}$$

Therefore, by an iterative procedure (that is, through successive decreases in the value of the functional), it is hoped that the minimum of I is approached to any desired degree of accuracy.

3. Variational Formulation

Suppose that nominal functions $x(t)$, $u(t)$ satisfying the differential constraint (2) and the initial conditions (3) are available.[3] Let $\tilde{x}(t)$, $\tilde{u}(t)$ denote varied functions satisfying the differential constraint (2) and the initial conditions (3). The varied functions are related to the nominal function as follows:

$$\tilde{x}(t) = x(t) + \delta x(t) \quad , \quad \tilde{u}(t) = u(t) + \delta u(t) \tag{8}$$

where $\delta x(t)$ and $\delta u(t)$ denote the perturbations of x and u about the nominal values. Note that, at the endpoints,

$$t_i = \text{given} \qquad \delta x_i = 0 \tag{9}$$

$$t_f = \text{given} \quad , \quad \delta x_f = \text{free} \tag{10}$$

To first order, the values of the varied functional and the nominal functional are related by

$$\tilde{I} \cong I + \delta I \tag{11}$$

where the first variation δI is given by

$$\delta I = \int_{t_i}^{t_f} (f_x^T \delta x + f_u^T \delta u)\, dt + (g_x^T \delta x)_f \tag{12}$$

Here, f_x denotes the gradient of the scalar function f with respect to the vector x, f_u the gradient of the scalar function f with respect to the vector u, and g_x the gradient of the

[3] These functions can be found by selecting u(t) arbitrarily and integrating Eq. (2) forward subject to the initial conditions (3).

scalar function g with respect to the vector x. These gradients are defined by

$$f_x = \begin{bmatrix} \partial f/\partial x^1 \\ \partial f/\partial x^2 \\ \vdots \\ \partial f/\partial x^n \end{bmatrix}, \quad f_u = \begin{bmatrix} \partial f/\partial u^1 \\ \partial f/\partial u^2 \\ \vdots \\ \partial f/\partial u^m \end{bmatrix} \tag{13}$$

and

$$g_x = \begin{bmatrix} \partial g/\partial x^1 \\ \partial g/\partial x^2 \\ \vdots \\ \partial g/\partial x^n \end{bmatrix} \tag{14}$$

Also to first order, Eq. (2) can be approximated by

$$\delta\dot{x} - \varphi_x^T \delta x - \varphi_u^T \delta u = 0 \tag{15}$$

Here, φ_x denotes the n x n matrix whose jth column is the gradient of the function φ^j with respect to the vector x and φ_u denotes the m x n matrix whose jth column is the gradient of the function φ^j with respect to the vector u. These matrices are defined as

$$\varphi_x = \begin{bmatrix} \partial\varphi^1/\partial x^1 & \partial\varphi^2/\partial x^1 & \cdots\cdots & \partial\varphi^n/\partial x^1 \\ \partial\varphi^1/\partial x^2 & \partial\varphi^2/\partial x^2 & \cdots\cdots & \partial\varphi^n/\partial x^2 \\ \cdots & \cdots & \cdots & \cdots \\ \partial\varphi^1/\partial x^n & \partial\varphi^2/\partial x^n & \cdots\cdots & \partial\varphi^n/\partial x^n \end{bmatrix} \tag{16}$$

and

$$
\varphi_u = \begin{bmatrix} \partial\varphi^1/\partial u^1 & \partial\varphi^2/\partial u^1 & \ldots\ldots & \partial\varphi^n/\partial u^1 \\ \partial\varphi^1/\partial u^2 & \partial\varphi^2/\partial u^2 & \ldots\ldots & \partial\varphi^n/\partial u^2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial\varphi^1/\partial u^m & \partial\varphi^2/\partial u^m & \ldots\ldots & \partial\varphi^n/\partial u^m \end{bmatrix} \tag{17}
$$

To first order, the greatest decrease in the value of the functional (11) is achieved if the first variation (12) is minimized. Here, we limit our analysis to those variations which satisfy the isoperimetric constraint

$$
K = \int_{t_i}^{t_f} \delta u^T \delta u \, dt \tag{18}
$$

where K is a prescribed positive constant.

4. Derivation of the Algorithm

We seek to determine the variations $\delta x(t)$, $\delta u(t)$ which minimize the functional (12) subject to the boundary conditions (9)-(10), the differential constraint (15), and the isoperimetric constraint (18). This variational problem of the Bolza type (see, for instance, Chapter 2 of Ref. 3) can be recast as that of minimizing the functional

$$I^* = \int_{t_i}^{t_f} F(\delta x, \delta\dot{x}, \delta u, \lambda, \alpha)dt + (g_x^T \delta x)_f \tag{19}$$

subject to (9)-(10), (15), (18). In the above expression, the fundamental function F is given by

$$F = f_x^T \delta x + f_u^T \delta u + \lambda^T(\delta\dot{x} - \varphi_x^T \delta x - \varphi_u^T \delta u) + (1/2\alpha)\, \delta u^T \delta u \tag{20}$$

where the vector λ denotes a variable Lagrange multiplier

$$\lambda = \begin{bmatrix} \lambda^1 \\ \lambda^2 \\ \vdots \\ \lambda^n \end{bmatrix} \tag{21}$$

and the scalar $1/2\alpha$ denotes a constant Lagrange multiplier. If one introduces the Hamiltonian

$$H = f - \lambda^T \varphi \tag{22}$$

and observes that

$$H_x = f_x - \varphi_x \lambda \quad , \quad H_u = f_u - \varphi_u \lambda \tag{23}$$

the fundamental function (20) can be rewritten as

$$F = H_x^T \delta x + H_u^T \delta u + \lambda^T \delta\dot{x} + (1/2\alpha)\delta u^T \delta u \tag{24}$$

4.1. <u>Euler Equations</u>. The optimum variations $\delta x(t)$, $\delta u(t)$ must satisfy the Euler equations

$$(d/dt)F_{\delta\dot{x}} = F_{\delta x} \quad , \quad 0 = F_{\delta u} \tag{25}$$

where $F_{\delta\dot{x}}$, $F_{\delta x}$, $F_{\delta u}$ respectively denote the gradients of the fundamental function F with respect to the vectors $\delta\dot{x}$, δx, δu. These gradients are given by

$$F_{\delta\dot{x}} = \begin{bmatrix} \partial F/\partial(\delta\dot{x}^1) \\ \partial F/\partial(\delta\dot{x}^2) \\ \vdots \\ \partial F/\partial(\delta\dot{x}^n) \end{bmatrix} , \quad F_{\delta x} = \begin{bmatrix} \partial F/\partial(\delta x^1) \\ \partial F/\partial(\delta x^2) \\ \vdots \\ \partial F/\partial(\delta x^n) \end{bmatrix} , \quad F_{\delta u} = \begin{bmatrix} \partial F/\partial(\delta u^1) \\ \partial F/\partial(\delta u^2) \\ \vdots \\ \partial F/\partial(\delta u^m) \end{bmatrix} \tag{26}$$

After observing that

$$F_{\delta\dot{x}} = \lambda \quad , \quad F_{\delta x} = H_x \quad , \quad F_{\delta u} = H_u + \delta u/\alpha \tag{27}$$

we see that Eqs. (25) yield the relations

$$\dot{\lambda} = H_x \quad , \quad \delta u = -\alpha H_u \tag{28}$$

which must be solved in combination with Eq. (15) and the boundary conditions (9)-(10). The control change δu is proportional to the gradient of the Hamiltonian with the respect to the control u; this is why the procedure is termed the gradient method. The quantity α is a scale factor for the control variation and, hence, is called the stepsize of the gradient method.

4.2. Transversality Condition. The boundary conditions are partly of the fixed endpoint type and partly of the variable endpoint type. The latter must be derived from the transversality condition

$$[(F - F^T_{\delta\dot{x}}\,\delta\dot{x})\delta t + F^T_{\delta\dot{x}}\,\delta(\delta x)]_i^f + [g_x^T\,\delta(\delta x)]_f = 0 \tag{29}$$

which must be satisfied for every system of variations consistent with the boundary conditions (9)-(10). Use of the boundary conditions and Eq. (27-1) allows one to rewrite (29) in the form

$$t_f = \text{given} \quad , \quad [(\lambda + g_x)^T\,\delta(\delta x)]_f = 0 \tag{30}$$

This equation is satisfied for every variation $\delta(\delta x_f)$ providing

$$t_f = \text{given} \quad , \quad (\lambda + g_x)_f = 0 \tag{31}$$

4.3. Summary of the Equations. The optimum corrections $\delta x(t)$, $\delta u(t)$ and the multiplier distribution $\lambda(t)$ are governed by Eqs. (15) and (28), which must be solved subject to the boundary conditions (9) and (31). Note that this differential system is linear. Because of the nature of the boundary conditions, the equations are uncoupled and need not be solved simultaneously. Specifically, it is convenient to integrate Eq. (28-1) backward subject to the final conditions (31) in order to obtain the multiplier distribution $\lambda(t)$. Once $\lambda(t)$ is known, the control change $\delta u(t)$ is supplied by Eq. (28-2), providing the stepsize α is specified. In turn, the state change $\delta x(t)$ is obtained by integrating Eq. (15) forward subject to the initial conditions (9).

The solution can be made independent of the stepsize α if one introduces the auxiliary variables

$$A = \delta x/\alpha \quad , \quad B = \delta u/\alpha \tag{32}$$

where A denotes an n-vector proportional to the state change and B denotes an m-vector proportional to the control change. With these variables, Eqs. (15) and (28) become

$$\dot{A} = \varphi_x^T A + \varphi_u^T B \quad , \quad \dot{\lambda} = H_x \quad , \quad B = - H_u \tag{33}$$

and must be solved subject to the boundary conditions

$$t_i = \text{given} \quad , \quad A_i = 0 \tag{34}$$

$$t_f = \text{given} \quad , \quad (\lambda + g_x)_f = 0 \tag{35}$$

Once the functions A(t) and B(t) are known, the corrections $\delta x(t)$ and $\delta u(t)$ are determined through Eqs. (32). Of course, this requires the specification of the stepsize α (see Section 5).

4.4. Relation between the Isoperimetric Constant and the Stepsize. If Eqs. (18) and (28-2) are combined, the following relation is obtained:

$$K = \alpha^2 \int_{t_i}^{t_f} H_u^T H_u \, dt \tag{36}$$

Since the integrand is known along a given nominal curve, Eq. (36) supplies a one-to-one correspondence between the isoperimetric constant K and the stepsize α. Therefore, one need not prescribe K and can reason directly on α, as in the considerations to follow.

4.5. Descent Property. If Eq. (15) is premultiplied by λ^T and integrated over the interval (t_i, t_f), one obtains the relation

$$0 = \int_{t_i}^{t_f} \lambda^T(\delta\dot{x} - \varphi_x^T \delta x - \varphi_u^T \delta u)dt \tag{37}$$

which, upon integration by parts, can be rewritten as

$$0 = (\lambda^T \delta x)_i^f - \int_{t_i}^{t_f} [(\dot{\lambda} + \varphi_x \lambda)^T \delta x + (\varphi_u \lambda)^T \delta u]dt \tag{38}$$

Upon adding (12) and (38) and accounting for Eqs. (9) and (23), we see that the first variation becomes

$$\delta I = \int_{t_i}^{t_f} H_u^T \delta u dt + \int_{t_i}^{t_f} (H_x - \dot{\lambda})^T \delta x dt + [(\lambda + g_x)^T \delta x]_f \tag{39}$$

We note that the second and third terms in Eq. (39) vanish because of the Euler equation (28-1) and the final condition (31-2). Therefore, Eq. (39) reduces to

$$\delta I = \int_{t_i}^{t_f} H_u^T \delta u \, dt \tag{40}$$

which, in the light of (28-2), can be rewritten as

$$\delta I = -\alpha \int_{t_i}^{t_f} H_u^T H_u \, dt \tag{41}$$

and, because of (36), is equivalent to

$$\delta I = -K/\alpha \tag{42}$$

Either of Eqs. (41) or (42) shows that the first variation is negative for $\alpha > 0$. Therefore, if α is sufficiently small, the functional (1) is bound to decrease. This descent property is the most important aspect of the gradient method.

5. Optimum Stepsize

The next step is to determine the optimum value of the parameter α. Clearly, this cannot be done by reasoning in terms of the first variation alone, since δI is linear in α. This being the case, we expand the functional (1) to second-order terms as follows:

$$\tilde{I} \cong I + \delta I + \frac{1}{2}\delta^2 I \tag{43}$$

where δI is the first variation and $\delta^2 I$ the second variation.

The first variation (12) can be taken into the alternate form (41), that is,

$$\delta I = -P\alpha \tag{44}$$

where P denotes the performance index

$$P = \int_{t_i}^{t_f} H_u^T H_u \, dt \tag{45}$$

Note that $P = 0$ for the exact variational solution ($H_u = 0$) and $P > 0$ for any other curve ($H_u \neq 0$). In view of (33-3), Eq. (45) can be rewritten as

$$P = \int_{t_i}^{t_f} B^T B \, dt \tag{46}$$

The second variation of the functional (1) is given by

$$\delta^2 I = \int_{t_i}^{t_f} (\delta x^T f_{xx} \delta x + 2\delta x^T f_{xu} \delta u + \delta u^T f_{uu} \delta u) \, dt + (\delta x^T g_{xx} \delta x)_f \tag{47}$$

and, in view of the definitions (32), can be rewritten as

$$\delta^2 I = Q\alpha^2 \tag{48}$$

where

$$Q = \int_{t_i}^{t_f} (A^T f_{xx} A + 2A^T f_{xu} B + B^T f_{uu} B)\, dt + (A^T g_{xx} A)_f \tag{49}$$

Because of (44) and (48), the varied functional (43) becomes

$$\tilde{I} = I - P\alpha + \frac{1}{2} Q\alpha^2 \tag{50}$$

The stepsize α which minimizes $\tilde{I}$ must satisfy the relation

$$d\tilde{I}/d\alpha = 0 \tag{51}$$

whose explicit form is the following:

$$-P + Q\alpha = 0 \tag{52}$$

and admits the solution

$$\alpha_o = P/Q \tag{53}$$

The significance of the quantities P and Q is clear from Eqs. (44) and (48): they are the values taken by the first variation and the second variation for $\alpha = 1$.

Remark 5.1. In practice, Eq. (53) must be replaced by

$$\alpha = \rho\mu\alpha_o = \rho\mu(P/Q) \tag{54}$$

where

$$\rho = \pm 1 \quad , \qquad 0 \le \mu \le 1 \tag{55}$$

respectively denote a direction factor and a scaling factor. The direction factor ρ is determined so that the first variation

$$\delta I = -\alpha P = -\ \rho\mu(P^2/Q) \tag{56}$$

is negative; this is precisely the case if one chooses

$$\rho = \text{sign } Q \tag{57}$$

with the implication that

$$\alpha = \mu(P/Q)\,\text{sign } Q = \mu P/|Q| \tag{58}$$

If the stepsize α is chosen according to (58) and if the scaling factor μ is in the range $0 \le \mu \le 1$, the integral (1) can be shown to decrease not only to first order but also to second order. The correct value of μ must be selected so that Ineq. (7) is satisfied even when expansions are not used, that is, even when the functional (1) is calculated exactly for both the nominal control $u(t)$ and the varied control $\tilde{u}(t)$; for this purpose, see Section 7.

6. Alternate Determination of the Optimum Stepsize

Alternatively, the optimum stepsize α can be determined by reasoning on the augmented functional

$$J = I + \int_{t_i}^{t_f} \lambda^T(\dot{x} - \varphi)\,dt \tag{59}$$

which, because of (1) and (22), is equivalent to

$$J = \int_{t_i}^{t_f} (H + \lambda^T\dot{x})\,dt + [g(x)]_f \tag{60}$$

To second order, the expansion of this functional is given by

$$\tilde{J} \cong J + \delta J + \frac{1}{2}\,\delta^2 J \tag{61}$$

where δJ is the first variation and $\delta^2 J$ the second variation.

The first variation of (60) can be written as

$$\delta J = \int_{t_i}^{t_f} (H_x^T\,\delta x + H_u^T\,\delta u + \lambda^T\,\delta\dot{x})dt + (g_x^T\,\delta x)_f \tag{62}$$

and, after an integration by parts is performed and Eqs. (9) are accounted for, becomes

$$\delta J = \int_{t_i}^{t_f} H_u^T\,\delta u dt + \int_{t_i}^{t_f} (H_x - \dot{\lambda})^T\,\delta x dt + [(\lambda + g_x)^T\,\delta x]_f \tag{63}$$

which is identical with (39). Therefore, one obtains

$$\delta J = \delta I \tag{64}$$

with the implication that

$$\delta J = - P\alpha \tag{65}$$

where

$$P = \int_{t_i}^{t_f} B^T B dt \tag{66}$$

Clearly, the first variation of the augmented functional is negative for $\alpha > 0$. Therefore, if α is sufficiently small, the augmented functional (60) is bound to decrease.

The second variation of (60) is given by

$$\delta^2 J = \int_{t_i}^{t_f} (\delta x^T H_{xx} \delta x + 2 \delta x^T H_{xu} \delta u + \delta u^T H_{uu} \delta u) dt + (\delta x^T g_{xx} \delta x)_f \tag{67}$$

and, in the light of Eqs. (32), is equivalent to

$$\delta^2 J = R\alpha^2 \tag{68}$$

where

$$R = \int_{t_i}^{t_f} (A^T H_{xx} A + 2A^T H_{xu} B + B^T H_{uu} B) dt + (A^T g_{xx} A)_f \tag{69}$$

Because of Eqs. (65) and (68), the varied augmented functional (61) reduces to

$$\tilde{J} = J - P\alpha + \frac{1}{2} R\alpha^2 \tag{70}$$

which is formally identical with (50), the only difference being that Q is replaced by R. With this understanding, the uncorrected optimum stepsize is given by Eq. (53) with Q replaced by R; the stepsize corrected to ensure the negativeness of the first variation as well as the negativeness of the total variation is given by Eq. (58) with Q replaced by R, that is, by

$$\alpha = \mu(P/R) \text{ sign } R = \mu P/|R| \tag{71}$$

Remark 6.1. If the functional (1) is linear in x and u, that is, if

$$f_{xx} = 0 \quad , \quad f_{xu} = 0 \quad , \quad f_{uu} = 0 \quad , \quad g_{xx} = 0 \tag{72}$$

the relation

$$Q = 0 \tag{73}$$

ensues from (49). Under these conditions, Eq. (52) no longer supplies an optimum value for α. Clearly, the algorithm of Section 5 fails, and the optimum value of α must be determined with the algorithm described in Section 6.

7. Summary of the Algorithm

In the light of the previous discussion, the complete algorithm can be summarized as follows:

(a) Choose a nominal control $u(t)$, obtain a consistent state $x(t)$ by integrating Eq. (2) forward subject to the initial conditions (3), and compute the functional I using Eq. (1).

(b) Determine the multiplier distribution $\lambda(t)$ by backward integration of (33-2) subject to the final conditions (35).

(c) Compute the function $B(t)$ using Eq. (33-3).

(d) Determine the function $A(t)$ by forward integration of (33-1) subject to the initial conditions (34).

(e) Compute the values of P and Q using Eqs. (46) and (49) or the values of P and R using Eqs. (66) and (69).

(f) Determine the stepsize α using Eq. (58) with $\mu = 1$ or Eq. (71) with $\mu = 1$.

(g) Compute the control change $\delta u(t)$ using Eq. (32-2).

(h) Determine the new control $\tilde{u}(t)$ using Eq. (8-2), obtain a consistent state $\tilde{x}(t)$ by forward integration of Eq. (2) subject to the initial conditions (3), and compute the functional $\tilde{I}$ using Eq. (1).

(i) If Ineq. (7) is satisfied, the scaling factor $\mu = 1$ is acceptable. If Ineq. (7) is violated, the scaling factor μ must be replaced by a smaller value; then, steps (f), (g), (h) must be repeated until Ineq. (7) is satisfied. The simplest way to generate smaller stepsizes is a bisection process: the value of μ is successively halved until satisfaction of Ineq. (7) occurs.

(j) After a value of α ensuring satisfaction of Ineq. (7) has been determined, the iteration is completed. Then, the varied control $\tilde{u}(t)$ becomes the nominal control $u(t)$ for the next iteration, and the procedure is repeated until a predetermined stopping condition is satisfied. For instance, one can require the satisfaction of the inequality

$$P \leq \epsilon \tag{74}$$

where P is the performance index defined by Eq. (45) and ϵ is a small number.

8. Numerical Examples

In order to illustrate the theory, two numerical examples are supplied. These examples have been studied previously by Jacobson in Refs. 4-5. For simplicity, all the symbols employed in this section denote scalar quantities.

Example 8.1. Consider the problem of minimizing the functional

$$I = \int_{t_i}^{t_f} (10x^2 + u^2)dt + 10x_f^2 \tag{75}$$

subject to the differential constraint

$$\dot{x} + 0.2x - 10\tanh u = 0 \tag{76}$$

and the boundary conditions

$$t_i = 0 \quad , \quad x_i = 5 \tag{77}$$

$$t_f = 0.5 \quad , \quad x_f = \text{free} \tag{78}$$

Assume the nominal control

$$u(t) = -0.5 \tag{79}$$

Starting with this nominal control, we employ the algorithm summarized in Section 7 in order to obtain the solution iteratively. The search technique described in Section 5 is used to find the optimum stepsize. The stopping condition (74) is employed with $\varepsilon = 10^{-4}$.

Computations were performed on the Rice University Burroughs B-5500 computer in double-precision arithmetic. The algorithm was programmed in extended ALGOL. The interval of integration was divided into 64 steps. The differential system (33)-(35) was integrated using the Adams-Bashforth/Adams-Moulton four-step predictor-corrector method (Ref. 6). The definite integrals I, P, Q were computed using Simpson's rule. The numerical results are presented in Figs. 1-2 and Table 1. Specifically, Fig. 1 shows the control history u(t) for different iterations N. Also, Table 1 presents the functions I(N) and P(N),where I is the functional (1) and P the performance index (45). As the analysis shows, convergence is quite rapid, in that Ineq. (74) is satisfied after 10 iterations. Note that the first four significant figures of the functional (1) did not change after 6 iterations, even though the algorithm kept producing small changes in the control.

Table 1

The functions I(N) and P(N)

N	I	P
0	123.44	0.11×10^{6}
1	48.20	0.14×10^{4}
2	41.82	0.73×10^{1}
3	41.72	0.29×10^{1}
4	41.62	0.99×10^{0}
5	41.60	0.29×10^{0}
6	41.59	0.13×10^{0}
7	41.59	0.68×10^{-2}
8	41.59	0.17×10^{-2}
9	41.59	0.26×10^{-3}
10	41.59	0.74×10^{-4}

Example 8.2. Consider the problem of minimizing the functional

$$I = \int_{t_i}^{t_f} (x^2 + u^2)\, dt \tag{80}$$

subject to the differential constraints

$$\dot{x} - y = 0 \quad , \quad \dot{y} + x - 1.4y + 0.14y^3 - 4u = 0 \tag{81}$$

and the boundary conditions

$$t_i = 0 \;, \quad x_i = -5 \;, \quad y_i = -5 \tag{82}$$

$$t_f = 2.5 \;, \quad x_f = \text{free} \;, \quad y_f = \text{free} \tag{83}$$

Assume the nominal control

$$u(t) = -0.5 \tag{84}$$

Starting with this nominal control, we employ the algorithm summarized in Section 7 in order to obtain the solution iteratively. The search technique described in Section 5 is used to find the optimum stepsize. The stopping condition (74) is employed with $\varepsilon = 10^{-4}$.

Computations were performed on the Rice University Burroughs B-5500 computer in double-precision arithmetic. The algorithm was programmed in extended ALGOL. The interval of integration was divided into 320 steps. The differential system (33)-(35) was integrated using the Adams-Bashforth/Adams-Moulton four-step predictor-corrector method (Ref. 6). The definite integrals I, P, Q were computed using Simpson's rule. The numerical results are presented in Figs. 3-5 and Table 2. Specifically, Fig. 3 shows the control history u(t) and Figs. 4-5 show the state history x(t), y(t) for different iterations N.

Also, Table 2 presents the functions I(N) and P(N), where I is the functional (1) and P the performance index (45). As the analysis shows, convergence is quite rapid, in that Ineq. (74) is satisfied after 15 iterations. Note that the first four significant figures of the functional (1) did not change after 9 iterations, even though the algorithm kept producing small changes in the control.

Table 2

The functions I(N) and P(N)

N	I	P
0	98.74	0.46×10^{4}
1	45.55	0.30×10^{3}
2	33.01	0.15×10^{2}
3	30.27	0.63×10^{1}
4	29.66	0.23×10^{1}
5	29.47	0.61×10^{0}
6	29.41	0.27×10^{0}
7	29.38	0.88×10^{-1}
8	29.38	0.37×10^{-1}
9	29.37	0.12×10^{-1}
10	29.37	0.53×10^{-2}
11	29.37	0.19×10^{-2}
12	29.37	0.81×10^{-3}
13	29.37	0.31×10^{-3}
14	29.37	0.12×10^{-3}
15	29.37	0.50×10^{-4}

9. Discussion and Conclusions

An analytical approach to the gradient method is presented within the framework of the Bolza problem of the calculus of variations. The first variation is minimized subject to the linearized differential constraint and an isoperimetric constraint on the control variation. Since the resulting Euler equations are linear, the differential system describing the optimum corrections is linear. The properties of this system are studied, and the solutions are related to the stepsize α.

Two search techniques are presented: (a) the optimum stepsize is obtained by minimizing the functional (1) expanded to second order or (b) the optimum stepsize is obtained by minimizing the augmented functional (59) expanded to second order. Technique (a) requires the second derivatives of the functions f and g, but not the second derivatives of the function φ; technique (b) requires the use of all the second derivatives. This being the case, procedure (a) requires less time per iteration than procedure (b) and, for that reason, has been employed in this paper. However, if the functional (1) is linear in x and u, procedure (a) fails to produce an optimum stepsize and, therefore, cannot be employed

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List of Captions

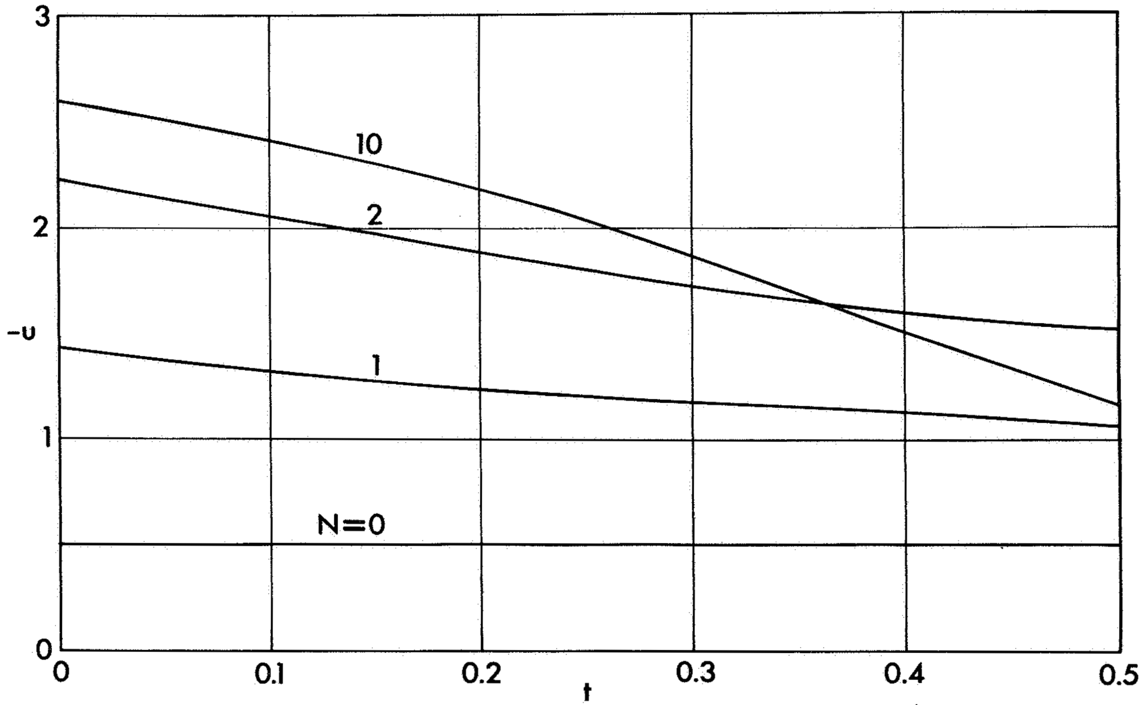

Fig. 1 The function u(t).

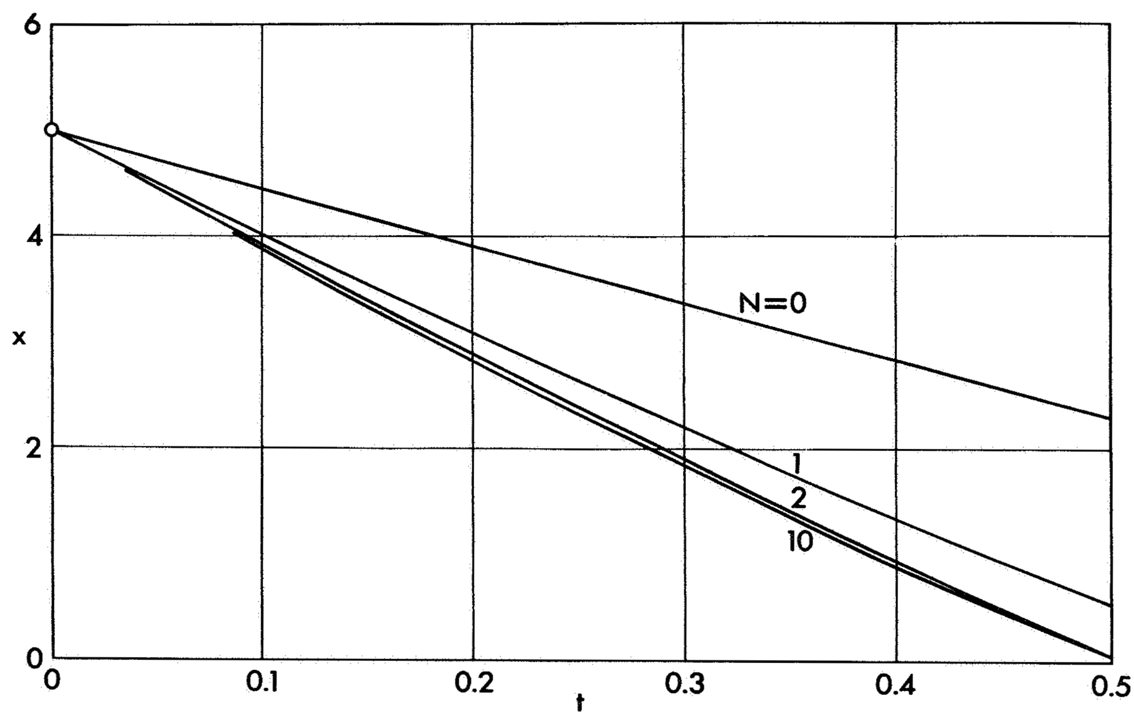

Fig. 2 The function x(t).

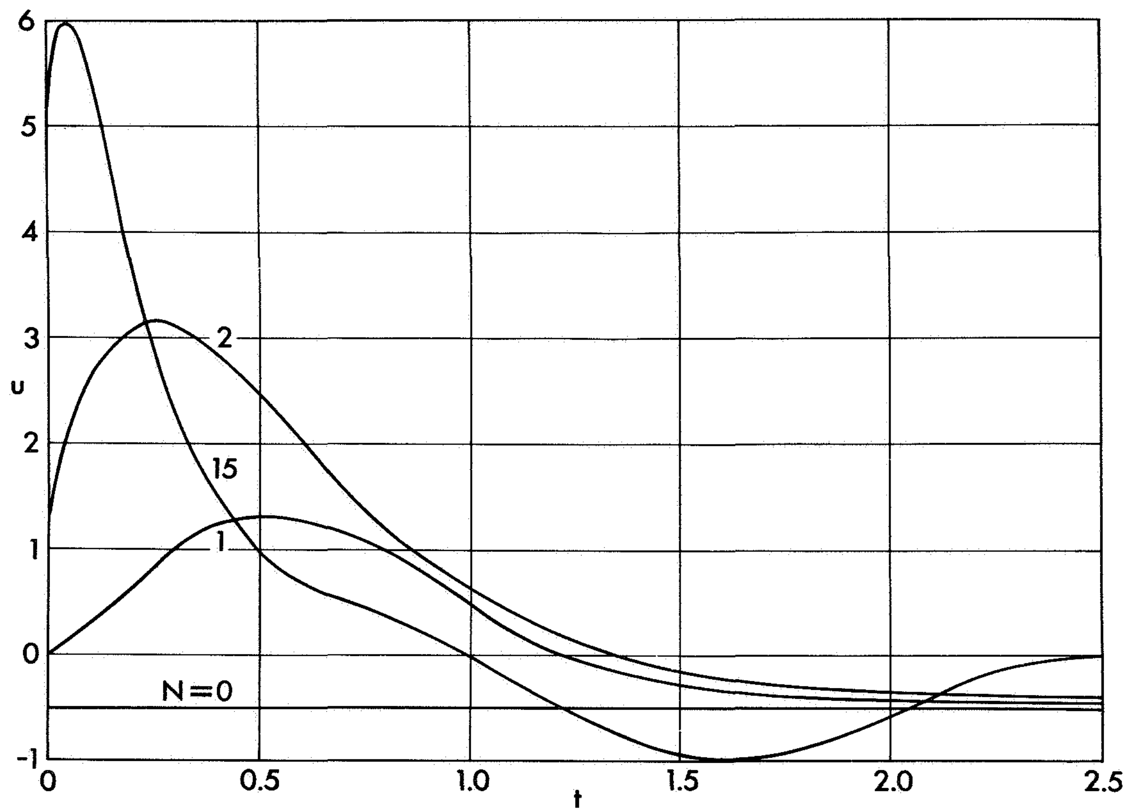

Fig. 3 The function u(t).

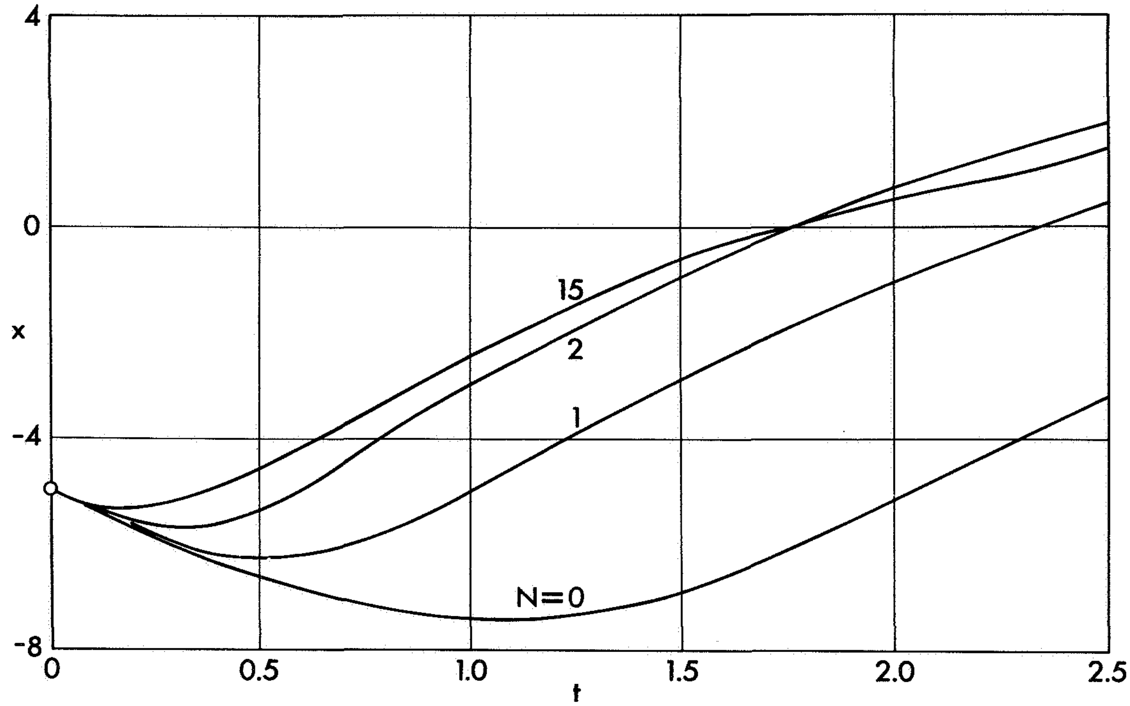

Fig. 4 The function x(t).

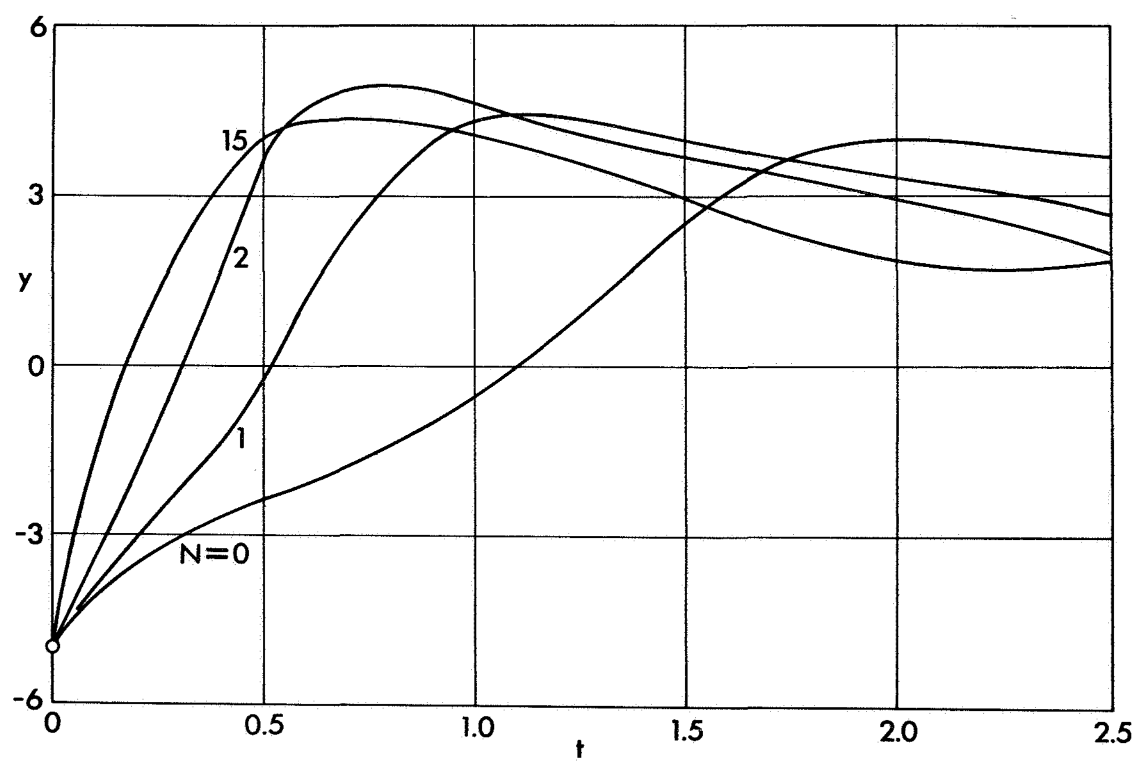

Fig. 5 The function y(t).